JOURNAL OF DIFFERENTIAL EQUATIONS **138**, 351–379 (1997)
ARTICLE NO. DE973279

Uniqueness Theorems for Positive Radial Solutions of Quasilinear Elliptic Equations in a Ball*

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Received March 5, 1997

We establish a new Pohozaev-type identity and use it to prove a theorem on the uniqueness of positive radial solutions to the quasilinear elliptic problem $\operatorname{div}(|\nabla u|^{m-2}\nabla u)+f(u)=0$ in B, and $u=0$ on ∂B, where B is a finite ball in $\mathbb{R}^n$, $n \geqslant 3$ and $1 < m < n$. Applying this main uniqueness result we can prove that the [illegible] plete answer to an open problem raised by Brezis and Nirenberg in 1983 in the case $n \geqslant 6$. We shall also derive some partial results to the open problem in the cases $n=3, 4, 5$.

1. INTRODUCTION

Let B be a finite ball in $\mathbb{R}^n$, $n \geqslant 3$. We are concerned with the problem of uniqueness of radial solutions to the quasilinear elliptic equation

$$\begin{aligned} &\operatorname{div}(|\nabla u|^{m-2}\nabla u)+f(u)=0 \quad \text{in } B, \\ &u>0 \quad \text{in } B, \qquad u=0 \quad \text{on } \partial B, \end{aligned} \tag{1.1$_m$}$$

where $1<m\leqslant n$, and ∇u denotes the gradient of u. When $m=2$, Equation (1.1_m) reduces to the well-known semilinear elliptic equation

$$\begin{aligned} &\Delta u+f(u)=0 \quad \text{in } B, \\ &u>0 \quad \text{in } B, \qquad u=0 \quad \text{on } \partial B, \end{aligned} \tag{1.1}$$

where Δ denotes the n-dimensional Laplacian. The uniqueness of nontrivial solutions to Problem (1.1) has received extensive investigation in the past decade, see in particular [6], [12], [14–15] and [22–23] when $f(u)$ is negative near $u=0$ and positive for large u; see also [1–3], [13], [16] and [27–29] when $f(u)$ is always positive for $u>0$.

* Research supported in part by NSERC-Canada.

0022-0396/97 $25.00

The symmetry of solutions to Problem (1.1) was studied in an elegant paper [26] of Serrin, where the powerful Alexandroff–Serrin method was developed. This device was extended and generalized by many authors, see, for example, Gidas *et al.* [10] and Zou [30]. In particular, it is proved in [10] that if f is of class C^1, then the solutions of (1.1) are necessarily radial.

The simplest and best understood example of (1.1) seems to be the case when $f(u)=u^p$, $p>1$. The semilinear equation

$$\Delta u+u^p=0 \qquad p>1, \tag{1.2}$$

is known as the Lane–Emden equation in astrophysics. In this context, when $n=3$, the function u represents the density of a single star. When $p=(n+2)/(n-2)$, (1.2) is also a special case of the Yamabe problem in differential geometry, and it is relevant to Yang–Mills equations for $n=4$. It is proved in [10] that when $1<p<(n+2)/(n-2)$, the problem

$$\begin{gathered} \Delta u+u^p=0 \quad \text{in } B, \\ u>0 \quad \text{in } B, \qquad u=0 \quad \text{on } \partial B, \end{gathered} \tag{1.3}$$

has a unique solution. However, when $p\geqslant(n+2)/(n-2)$, a powerful identity of Pohozaev [24] demonstrates that (1.3) has no solutions at all (radial or nonradial) even if B is replaced by any starshaped bounded domain. The exponent $p=(n+2)/(n-2)$ sets up a dividing number for the existence and nonexistence of Problem (1.3). It is critical from the point of view of Sobolev embedding, since $p+1=2n/(n-2)$ is the limiting Sobolev exponent for the embedding $H_0^1(B)\subset L^{p+1}(B)$. This embedding is not compact when $p\geqslant(n+2)/(n-2)$.

The situation becomes very different and much more complicated if the non-linearity f does not have a constant growth. Erbe and Tang [7–8] proved that Problem (1.1) has a unique solution if f has a supercritical growth near zero and a subcritical growth near infinity, and if the growth order of $f(u)$ is a decreasing function of u. A model case was given by $f(u)=u^p$, for $u>1$; $=u^q$, for $0\leqslant u\leqslant 1$ and $1<p<(n+2)/(n-2)<q$.

In general, the uniqueness problem of (1.1) is extremely difficult to study even if the nonlinearity is of simple form. Let $\lambda>0$ and $\mu>0$ be constants and consider the problems

$$\begin{gathered} \Delta u+\lambda u+u^q=0 \quad \text{in } B, \\ u>0 \quad \text{in } B, \qquad u=0 \quad \text{on } \partial B, \end{gathered} \tag{1.4}$$

and

$$\begin{gathered} \Delta u + \mu u^p + u^q = 0 \quad \text{in } B, \\ u > 0 \quad \text{in } B, \qquad u = 0 \quad \text{on } \partial B. \end{gathered} \tag{1.5}$$

Let $q = (n+2)/(n-2)$, and $1 < p < q$. Then Brezis and Nirenberg [3] proved that

(i) When $n \geqslant 4$, Problem (1.4) has a solution if and only if $\lambda \in (0, \lambda_1)$, where λ_1 denotes the first eigenvalue of $-\Delta$; while Problem (1.5) has a solution for every $\mu > 0$.

(ii) When $n = 3$, Problem (1.4) has a solution if and only if $\lambda \in (\lambda_1/4, \lambda_1)$; while Problem (1.5) has a solution only for large values of $\mu > 0$.

Moreover, it is suggested in [3] by numerical computations, and later proved by Atkinson and Peletier [2], that

(iii) when $n = 3$, $q = 5$, and $1 < p < 3$, there is some $\mu_0 > 0$ such that Problem (1.5) has at least two solutions for $\mu > \mu_0$, a unique solution for $\mu = \mu_0$, and no solution for $\mu < \mu_0$.

An interesting open problem raised by Brezis and Nirenberg [3] was *whether or not the solution of* (1.4)–(1.5), *whose existence is ensured in* (i) *and* (ii), *is unique* (*except the case* $n = 3$, $q = 5$, *and* $1 < p < 3$ *in* (1.5)).

The uniqueness of solutions of Problem (1.4) was proved by Kwong and Li [13] for the range $1 < q < (n+2)/(n-2)$, and by Zhang [28] for $1 < q \leqslant (n+2)/(n-2)$. Some alternative proofs were provided by Srikanth [27] and Adimurthi and Yadava [1]. On the other hand, in view of Assertion (iii) above, one cannot expect that the uniqueness of solutions of Problem (1.5) holds for all $n \geqslant 3$ and $1 < p < q \leqslant (n+2)/(n-2)$. Very recenctly, Zhang [29] proved that (1.5) admits at most one solution if

$$(q-1)/(p+1) \leqslant 2/n. \tag{1.6}$$

For example, when $n = 3$ and $p = 3$, (1.6) is satisfied if $3 < q \leqslant 11/3$. In a more general setting, Ni and Nussbaum [16] studied the uniqueness problem of (1.1) for more general nonlinearities. Their results reveal that Problem (1.5) has a unique solution if

$$1 < p < q < n/(n-2). \tag{1.7}$$

For instance, when $n = 3$, (1.7) is satisfied if $1 < p < q < 3$. Note that both (1.6) and (1.7) cover only a part of the range $1 < p < q \leqslant (n+2)/(n-2)$ and exclude the most important case $q = (n+2)/(n-2)$.

As an important application of the main result of this paper, we shall prove

THEOREM 1. *Let* $\mu > 0$ *and* $1 \leqslant p < q \leqslant (n+2)/(n-2)$. *Then problem* (1.5) *has a unique solution when* $n \geqslant 6$.

Thus, the open problem concerning the uniqueness of (1.5) is completely solved for $n \geqslant 6$. We also get some partial results for the cases $n = 3, 4, 5$; for example, when $n = p = 3$, Problem (1.5) has at most one solution if $3 < q < 4.7748332$.

For a more general nonlinearity f, we shall assume that

(F1) $f \in C^1([0, \infty))$, and $f(0) = 0$.

(F2) $0 < (m-1)\, f(s) < sf'(s)$, for $s > 0$.

A positive radial solution of (1.1_m) satisfies

$$\left[(m-1)u'' + \frac{n-1}{t}u'\right]|u'|^{m-2} + f(u) = 0,$$

$$u(0) = \alpha > 0, \qquad u'(0) = 0.$$

It follows from Propositions A1–A4 of Franchi, Lanconelli and Serrin [9] that if $f(u)$ satisfies (F1), then this Cauchy problem has a unique solution which depends continuously on the initial data α. We denote the solution by $u(t, \alpha)$. Furthermore, if $f(u)$ satisfies (F1) and (F2), then $u'(t, \alpha) < 0$ in $(0, t_0)$ as long as $u(t, \alpha) > 0$ on this interval (see (iv) of Proposition 2.1 below). Thus the inverse of $u(t, \alpha)$, denoted by $t = t(u, \alpha)$, is well-defined and is also strictly decreasing in $(0, \alpha)$. As in [9] (page 223), $t = t(u, \alpha)$ is a solution of the equation

$$(m-1)\, t'' = \frac{n-1}{t}\, t'^2 + f(u)\, |t'|^m\, t', \tag{1.8}$$

where $' = d/du$.

Let $F(s)$, $\bar{H}(s)$, and $H(s)$ be defined by $F(s) = \int_0^s f(\tau)\, d\tau$,

$$\bar{H}(s) = (n-m)\, sf(s) - nmF(s), \tag{1.9}$$

and

$$H(s) = \begin{cases} \bar{H}(s)/f(s), & u > 0, \\ 0, & u = 0. \end{cases} \tag{1.10}$$

Then the following identity holds.

PROPOSITION 2. *If $u \in (0, \alpha)$, then*

$$P(u) = \int_{\alpha}^{u} H'(s)\, t^{n-1}/(t'\, |t'|^{m-2})\, ds, \tag{1.11}$$

where $' = d/du$, and

$$P(u) = [H(u) - (n-m)u]\, \frac{t^{n-1}}{t'\, |t'|^{m-2}} - (m-1)\, \frac{t^n}{|t'|^m} - mF(u)\, t^n. \tag{1.12}$$

Applying this proposition and the separation techniques employed by Peletier and Serrin [22–23] and Franchi *et al.* (Section 3.3 of [9]), we shall prove

MAIN THEOREM. *Assuming* (F1) *and* (F2) *hold and $H'(s) \leqslant 0$ for $s > 0$, then Problem* (1.1_m) *admits at most one radial solution.*

It is worth noting that H' in this theorem cannot be replaced with any of the functions $\bar{H}$, $\bar{H}'$ or H. This can be verified by the nonuniqueness example of [2] (see Assertion (iii) above). Note also that for the typical model $f = u^p$, $p > m-1$, $H'(u)$ is simply a constant $n - m - nm/(p+1)$.

Equation (1.1_m) was also studied by Franchi *et al.* [9], who proved the uniqueness of radial ground states for non-linearities having sublinear behavior for large u. By using the separation theorems of Section 3.3 of [9], Citti [5] was able to show that the method introduced by Kwong and Zhang [14] can be applied to (1.1_m), and some uniqueness results were obtained in the case $1 < m \leqslant 2$ and $n \geqslant 2 - 1/m$ for $f(u)$ which are similar to those studied in [14]. Also, a uniqueness result for radial solutions of (1.1_m) was proved by Adimurthi and Yadava [1] for $f(u) = \lambda u + u^q$, where $m - 1 < q < (nm - n + m)/(n - m)$.

This paper is organized as follows. In Section 2, we shall recall some known results on the radial solutions and classify the solutions into three types. Following an idea of Kawano *et al.* [11], we give characteristic descriptions for each type of solution. The main theorem and Theorem 1 will be proved in Section 3 and Section 4, respectively. In Section 5, we give an example to show that when (F2) is not assumed, the result of the main theorem is no longer valid.

2. GENERAL PROPERTIES OF RADIAL SOLUTIONS

We shall maintain the assumption (F1) throughout the remainder of the paper, without further mention. Recall from the result of [10] that any solution $u(x)$ of Problem (1.1) is radial for an appropriate choice of origin. Let $t=|x|$, then $u(x)=u(t)$ and

$$
\begin{aligned}
&u''+\frac{n-1}{t}u'+f(u)=0,\\
&u(0)=\alpha>0, \qquad u'(0)=0.
\end{aligned}
\tag{2.1}
$$

As noted in Section 1, a positive radial solution of (1.1_m) satisfies

$$
\begin{aligned}
&\left[(m-1)u''+\frac{n-1}{t}u'\right]|u'|^{m-2}+f(u)=0,\\
&u(0)=\alpha>0, \qquad u'(0)=0.
\end{aligned}
\tag{2.1_m}
$$

For a given $\alpha>0$, we shall denote the unique solution to problem (2.1) or (2.1_m) by $u(t,\alpha)$. The main purpose of this section is to present some fundamental properties of $u(t,\alpha)$ and to characterize the set of solutions.

PROPOSITION 2.1. *Let $u(t,\alpha)$ be a solution of Problem (2.1_m) and define*

$$b(\alpha)=\sup\{T\colon u(t,\alpha) \text{ is defined and } u(t,\alpha)>0 \text{ in } [0,T)\}. \tag{2.2}$$

Then we have

(i) $u(b(\alpha),\alpha)=0$ *when* $b(\alpha)<\infty$.

(ii) $u(t,\alpha)\in C^2(0,b(\alpha)]\cap C^1[0,b(\alpha)]$ *when* $b(\alpha)<\infty$, *or* $u(t,\alpha)\in C^2(0,\infty)\cap C^1[0,\infty)$ *when* $b(\alpha)=\infty$.

(iii) $u(t,\alpha)$ *is uniquely determined by* α. *Moreover, let* $u=u(t,\alpha)$ *and* $\hat{u}=u(t,\hat{\alpha})$, *with* $\alpha>0$, $\hat{\alpha}>0$, *be two solutions of* (2.1_m). *If there exists* $t_0\in[0,\min\{b(\alpha),b(\hat{\alpha})\}]$, *or* $t_0\in[0,\infty)$ *when* $b(\alpha)=b(\hat{\alpha})=\infty$, *such that* $u(t_0)=\hat{u}(t_0)$ *and* $u'(t_0)=\hat{u}'(t_0)$, *then* $u\equiv\hat{u}$.

(iv) $u'(t,\alpha)<0$ *in* $(0,b(\alpha)]$ *when* $b(\alpha)<\infty$, *or in* $(0,\infty)$ *otherwise.*

(v) $u(t,\alpha)\to 0$ *and* $u'(t,\alpha)\to 0$ *as* $t\to\infty$, *when* $b(\alpha)=\infty$.

Proof. (i) follows from (ii) and (iv). (ii) was obtained by Ni and Serrin in the Appendix of [18], (iii) was proved by Franchi, Lanconelli, and Serrin in the Appendix of [9], and (iv) follows from Proposition 1.2.6 of [9]. We only give a detailed proof of (v) here.

Let $\alpha>0$ be such that $u(t,\alpha)>0$ in $[0,\infty)$. By (iv) it follows that there is some u_∞ such that $0\leqslant u_\infty<\infty$ and $\lim_{t\to\infty}u(t,\alpha)=u_\infty$. Differentiating the energy function

$$E(t)=E(t,\alpha)=\frac{m-1}{m}|u'|^m+F(u), \tag{2.3}$$

one has $dE(t)/dt = -(n-1)/t\,|u'|^m$. Thus $E(t)$ is decreasing whenever $u(t)$ is defined and $E(t, \alpha)$ tends to a nonnegative constant as $t \to \infty$. Because of the convergence of $u(t, \alpha)$ and the continuity of $F(u)$, we conclude that $|u'(t, \alpha)|^m$ is convergent, and so also $u'(t, \alpha)$. Due to the fact that $u(t, \alpha) > 0$ on $[0, \infty)$, one must have

$$\lim_{t \to \infty} u'(t, \alpha) = 0.$$

It remains to show that $u_\infty = 0$. Suppose to the contrary that $u_\infty > 0$. Letting $t \to \infty$ in the equation of (2.1_m) we have

$$\lim_{t \to \infty} u''(t, \alpha)\,|u'(t, \alpha)|^{m-2} = -f(u_\infty)/(m-1) < 0. \tag{2.4}$$

It is readily seen that (2.4) implies

$$\lim_{t \to \infty} u''(t, \alpha) = -\infty \qquad \text{when} \quad m > 2,$$

$$\lim_{t \to \infty} u''(t, \alpha) = -f(u_\infty) < 0 \qquad \text{when} \quad m = 2.$$

They are not compatible with $\lim_{t \to \infty} u'(t, \bar{\alpha}) = 0$. If $1 < m < 2$, then there exist some real numbers $\iota > 0$ and $T_\iota > 0$ such that if $t > T_\iota$, then

$$u''(t, \alpha)/u'(t, \alpha) > \iota.$$

Integrating both sides of this inequality over $[T_\iota, t)$ and letting $t \to \infty$ yield

$$\lim_{t \to \infty} |u'(t, \alpha)| = \infty,$$

which contradicts $\lim_{t \to \infty} u'(t, \alpha) = 0$. Thus we obtain $\lim_{t \to \infty} u(t, \alpha) = 0$ as desired. ∎

PROPOSITION 2.2 (Ni–Pucci–Serrin). *Let $u = u(t, \alpha)$ be a solution of (2.1_m). Then*

$$\int_0^t \left\{ auf(u) - nF(u) + |u'|^m \left(\frac{n}{m} - a - 1 \right) \right\} \tau^{n-1}\, d\tau$$

$$= -au'(t)\,|u'(t)|^{m-2}\, u(t)\, t^{n-1} - \left(1 - \frac{1}{m}\right) |u'(t)|^m\, t^n - F(u(t))\, t^n, \tag{2.5}$$

where a is any real number.

For the proof of (2.5), see Ni and Serrin [17], [18] and Pucci and Serrin [25]. Let $a = n/m - 1$ in (2.5). Then we have

LEMMA 2.3. *Let $u = u(t, \alpha)$ be a solution of* (2.1_m). *Then*

$$\bar{P}(t) = \int_0^t \bar{H}(u(\tau))\, \tau^{n-1}\, d\tau, \tag{2.6}$$

where $\bar{H}(u)$ is defined in (1.9) *and*

$$\begin{aligned} \bar{P}(t) &= \bar{P}(t, \alpha) = \bar{P}(t, \alpha, u(t, \alpha)) \\ &= -(n-m)\, u'(t)\, |u'(t)|^{m-2}\, u(t)\, t^{n-1} - (m-1)\, |u'(t)|^m\, t^n - mF(u(t))\, t^n. \end{aligned} \tag{2.7}$$

Next we give a proof of identity (1.11) by starting with (2.7). Note that (1.11) can also be proved by a straightforward way by differentiating both sides of the identity with respect to u.

Proof of Proposition 2. By using the fact that $t_u = 1/u_t$, we have

$$\begin{aligned} &-(n-m)\, \frac{u t^{n-1}}{t'\, |t'|^{m-2}} - (m-1)\, \frac{t^n}{|t'|^m} - mF(u)\, t^n \\ &\quad = \int_\alpha^u \left[(n-m)\, sf(s) - mnF(s) \right] t^{n-1} t'\, ds. \end{aligned} \tag{2.8}$$

Recall from (1.8) that

$$f(u)\, t' = \left((m-1)\, t'' - \frac{n-1}{t}\, t'^2 \right) \Big/ |t'|^m.$$

We can evaluate the right side of (2.8), using integration by parts, as follows:

$$\begin{aligned} &\int_\alpha^u \left[(n-m)\, sf(s) - mnF(s) \right] t^{n-1} t'\, ds \\ &\quad = \int_\alpha^u H(s)\, f(s)\, t^{n-1} t'\, ds \\ &\quad = \int_\alpha^u H(s)\, t^{n-1} \left((m-1)\, t'' - \frac{n-1}{t}\, t'^2 \right) \Big/ |t'|^m\, ds \\ &\quad = \int_\alpha^u H(s)\, t^{n-1}\, \frac{(m-1)\, t''}{|t'|^m}\, ds - (n-1) \int_\alpha^u H(s)\, t^{n-2}/|t'|^{m-2}\, ds \\ &\quad = -\int_\alpha^u H(s)\, t^{n-1} d\left(\frac{1}{t'\, |t'|^{m-2}} \right) - (n-1) \int_\alpha^u H(s)\, t^{n-2}/|t'|^{m-2}\, ds \end{aligned}$$

$$= H(u)\, t^{n-1}/(t'\,|t'|^{m-2}) + \int_{\alpha}^{u} H'(s)\, t^{n-1}/(t'\,|t'|^{m-2})\, ds$$

$$+ (n-1)\int_{\alpha}^{u} H(s)\, t^{n-2}/|t'|^{m-2}\, ds - (n-1)\int_{\alpha}^{u} H(s)\, t^{n-2}/|t'|^{m-2}\, ds$$

$$= -H(u)\, t^{n-1}/(t'\,|t'|^{m-2}) + \int_{\alpha}^{u} H'(s)\, t^{n-1}/(t'\,|t'|^{m-2})\, ds.$$

Hence (1.11) follows. ∎

In the remainder of this section, we are concerned with some characteristic description for the solutions of (2.1_m).

DEFINITION 2.4. A solution $u(t, \alpha)$ is called a *crossing solution* if $b(\alpha) < \infty$, $u(t, \alpha) > 0$ in $[0, b(\alpha))$, and $u(b(\alpha), \alpha) = 0$. It is called a *decaying solution* if $u(t, \alpha) > 0$ in $[0, \infty)$ and $\lim_{t\to\infty} u(t, \alpha) = 0$.

It follows from Proposition 2.1(i) and (v) that every solution $u(t, \alpha)$ is either a crossing solution or a decaying solution. Before we start investigating the asymptotic behavior of decaying solutions, we digress for a moment to the special case $n = m$. We shall show that, in this case, every solution $u(t, \alpha)$, $\alpha > 0$ is necessarily a crossing solution, and therefore no decaying solutions exist. To our knowledge, this interesting result has not been observed in the current literature.

PROPOSITION 2.5. *If $n = m$, then $u(t, \alpha)$ is a crossing solution for each $\alpha > 0$. Consequently, (2.1_m) possesses no decaying solution.*

Proof. Let $n = m$, and suppose for contradiction that $u = u(t, \alpha)$, $\alpha > 0$ is a decaying solution of (2.1_m). Then $u' < 0$ in $(0, \infty)$ and we can show that

$$\lim_{t\to\infty} tu' = 0. \tag{2.9}$$

In fact, from (2.1_m) we can deduce

$$(m-1)u'' + \frac{n-1}{t}u' < 0, \qquad t \in (0, \infty),$$

which in turn implies that

$$(tu')' < 0, \qquad t \in (0, \infty),$$

because $n = m$. Thus tu' is strictly decreasing in $(0, \infty)$, and there is some number $\iota \leqslant 0$ such that $\lim_{t\to\infty} tu' = \iota$. If $\iota < 0$, then one can easily demonstrate that $\lim_{t\to\infty} u = -\infty$ to yield a contradiction. Hence $\iota = 0$ and (2.9) is proved.

By (2.9) and the fact that tu' is decreasing in $(0, \infty)$, we see that $tu' > 0$ in $(0, \infty)$, which contradicts $u' < 0$. The proof is completed. ∎

LEMMA 2.6. *Let* $1 < m < n$, *and let* $u(t, \alpha)$ *be a decaying solution of* (2.1_m), *then* $t^{(n-m)/(m-1)}u(t, \alpha)$ *is strictly increasing in* $[0, \infty)$.

Proof. Let $1 < m < n$, and $u = u(t, \alpha)$ be a decaying solution of (2.1_m). Then a straightforward calculation yields

$$(t^{(n-m)/(m-1)}u)' = t^{((n-m)/(m-1))-1}\left(\frac{n-m}{m-1}u + tu'\right), \qquad t > 0.$$

It suffices to show that

$$\frac{n-m}{m-1}u + tu' > 0 \qquad \text{for} \quad t > 0. \tag{2.10}$$

Note that

$$\left(\frac{n-m}{m-1}u + tu'\right)' = \frac{t}{m-1}\left((m-1)u'' + \frac{n-1}{t}u'\right) < 0.$$

Hence $(n-m)/(m-1)u + tu'$ is decreasing on $(0, \infty)$ and therefore $\lim_{t\to\infty}((n-m)/(m-1)u + tu') = 0$. Thus (2.10) follows and the proof is completed. ∎

DEFINITION 2.7. A decaying solution is called a *fast decaying solution* or a *ground state solution* if

$$\lim_{t\to\infty} t^{(n-m)/(m-1)}u(t, \alpha) \text{ exists and is finite,} \tag{2.11}$$

and a *slowly decaying solution* if

$$\lim_{t\to\infty} t^{(n-m)/(m-1)}u(t, \alpha) = \infty. \tag{2.12}$$

It is easily seen that every solution $u(t, \alpha)$ is classified into one of three types: *a crossing solution, a slowly decaying solution, or a ground state solution* (*a fast decaying solution*). Inspired by Kawano *et al.* [11], we can characterize each set of solutions as follows.

LEMMA 2.8. *Let* $\bar{P}(t, \alpha)$ *be as in* (2.7), *and define*

$$\bar{P}_\alpha = \limsup_{t\to b(\alpha)^-} P(t, \alpha). \tag{2.13}$$

(i) *If $u(t, \alpha)$ is a crossing solution, then*

$$\bar{P}_\alpha = \bar{P}(b(\alpha), \alpha) < 0. \tag{2.14}$$

(ii) *If $u(t, \alpha)$ is a ground state solution, and $f(u)$ satisfies*

$$\text{(F3)} \quad \limsup_{s \to 0^+} \frac{F(s)}{s^{\varepsilon_m}} = 0, \qquad \textit{where} \quad \varepsilon_m = \frac{n(m-1)}{n-m} > 0.$$

Then

$$\bar{P}_\alpha = \lim_{t \to \infty} \bar{P}(t, \alpha) = 0. \tag{2.15}$$

(iii) *If $u(t, \alpha)$ is a slowly decaying solution, and $f(u)$ satisfies*

$$\text{(F2}'\text{)} \quad 0 < (m-1) f(s) < sf'(s), \quad \textit{for} \ s > 0$$
$$\textit{and} \qquad \liminf_{u \to 0^+} uf'(u)/f(u) > m-1.$$

Then

$$\bar{P}_\alpha \geqslant 0, \tag{2.16}$$

and for any $T' > 0$, there exists $T'' > T'$ such that

$$\bar{P}(T'', \alpha) > 0. \tag{2.17}$$

Proof. (i) The proof of (2.14) is trivial, we omit it.

(ii) As observed in Section 3.2 of [9], if $u(t, \alpha)$ is a ground state solution and

$$\lim_{t \to \infty} t^{(n-m)/(m-1)} u(t, \alpha) = c_\alpha. \tag{2.18}$$

with $0 < c_\alpha < \infty$, then one can use the L'Hospital's rule to get

$$\lim_{t \to \infty} t^{(n-1)/(m-1)} \, |u'(t, \alpha)| = \frac{n-m}{m-1} \, c_\alpha. \tag{2.19}$$

Since $(n-1)/(m-1) > 1$, (2.19) implies

$$\lim_{t \to \infty} tu'(t, \alpha) = 0. \tag{2.20}$$

Now (2.15) follows readily from (F3) and (2.18–2.20).

(iii) Let $u(t, \alpha)$ be a slowly decaying solution. In view of assumption (F2′) we may pick some number $\bar{m}$, $\bar{m} > m$ such that

$$\frac{sf'(s)}{f(s)} > \bar{m} - 1, \qquad 0 < s < \alpha. \tag{2.21}$$

We can use this inequality to estimate $F(u)/uf(u)$. Note that when $0 < u < \alpha$,

$$\begin{aligned}\frac{F(u)}{uf(u)} &= \frac{\int_0^u f(s)\,ds}{uf(u)}\\ &= \frac{uf(u) - \int_0^u sf'(s)\,ds}{uf(u)} = 1 - \frac{\int_0^u sf'(s)\,ds}{uf(u)}\\ &< 1 - (\bar{m} - 1)\,\frac{\int_0^u f(s)\,ds}{uf(u)} = 1 - (\bar{m} - 1)\,\frac{F(u)}{uf(u)}.\end{aligned}$$

This gives

$$\frac{F(u(t, \alpha))}{u(t, \alpha)\, f(u(t, \alpha))} < \frac{1}{\bar{m}}. \tag{2.22}$$

For simplicity of notations, let $\xi = m/\bar{m}$. Then

$$0 < \xi < 1, \qquad mF(u(t, \alpha)) < \xi u(t, \alpha)\, f(u(t, \alpha)), \qquad t > 0. \tag{2.23}$$

Now, let $u = u(t, \alpha)$ and $t > 0$, we have

$$\begin{aligned}\bar{P}(t, \alpha) &= -(n-m)\, t^{n-1} uu'\, |u'|^{m-2} - (m-1)\, t^n\, |u'|^m - mt^n F(u)\\ &\geqslant -(n-m)\, t^{n-1} uu'\, |u'|^{m-2} - (m-1)\, t^n\, |u'|^m - \xi t^n uf(u)\\ &= -(m-1)\, t^{n-(n-m)/(m-1)} u'\, |u'|^{m-2}\\ &\quad \times \left[\frac{n-m}{m-1}\, t^{(n-m)/(m-1)-1} u + t^{(n-m)/(m-1)} u'\right]\\ &\quad + \xi tu(t^{n-1} u'\, |u'|^{m-2})'\\ &= -[(m-1)\, t^{n-(n-m)/(m-1)} u'\, |u'|^{m-2}\, (t^{(n-m)/(m-1)} u)'\\ &\quad + \xi tu(t^{n-1}\, |u'|^{m-1})']\\ &= t^n u\, |u'|^{m-1} \left[(m-1)\, \frac{(t^{(n-m)/(m-1)} u)'}{t^{(n-m)/(m-1)} u} - \xi\, \frac{(t^{n-1}\, |u'|^{m-1})'}{t^{n-1}\, |u'|^{m-1}}\right]\\ &= t^n u\, |u'|^{m-1}\, [(m-1) \ln(t^{(n-m)/(m-1)} u) - \xi \ln(t^{n-1}\, |u'|^{m-1})]'\end{aligned}$$

$$\begin{aligned}
&= t^n u\,|u'|^{m-1}\,[(m-1)\ln(t^{(n-m)/(m-1)}u) \\
&\quad -(m-1)\,\xi\ln(t^{(n-m)/(m-1)}u) \\
&\quad +(m-1)\,\xi\ln(t^{(n-m)/(m-1)}u)-\xi\ln(t^{n-1}\,|u'|^{m-1})]' \\
&= t^n u\,|u'|^{m-1}\,[(m-1)\ln(t^{(n-m)/(m-1)}u) \\
&\quad -(m-1)\,\xi\ln(t^{(n-m)/(m-1)}u) \\
&\quad +\xi\ln(t^{(n-m)/(m-1)}u)^{m-1}-\xi\ln(t^{(n-m)/(m-1)}\,|u'|)^{m-1}]' \\
&= t^n u\,|u'|^{m-1}\,[(m-1)(1-\xi)\ln(t^{(n-m)/(m-1)}u) \\
&\quad +\xi\ln((t^{(n-m)/(m-1)}u)/(t^{(n-1)/(m-1)}\,|u'|))^{m-1}]' \\
&= t^n u\,|u'|^{m-1}\,[(m-1)(1-\xi)\ln(t^{(n-m)/(m-1)}u) \\
&\quad +\xi\ln(u/(t\,|u'|))^{m-1}]'.
\end{aligned}$$

Recall that $\lim_{t\to\infty} t^{(m-1)/(n-m)}u = +\infty$, and

$$u/(t\,|u'|) \geqslant \frac{m-1}{n-m}, \qquad \text{(by (2.10))}.$$

We have

$$\lim_{t\to\infty}\,[(m-1)(1-\xi)\ln(t^{(n-m)/(m-1)}u)+\xi\ln(u/(t\,|u'|))^{m-1}] = +\infty.$$

Thus for any $T'>0$, there exists a $T''>T'$ such that

$$[(m-1)(1-\xi)\ln(t^{(n-m)/(m-1)}u)+\xi\ln(u/(t\,|u'|))^{m-1}]'|_{t=T''}>0.$$

Now (2.16) and (2.17) follow immediately. The proof is completed. ∎

As an important application of Lemma 2.8, we present in the next theorem some criteria for the existence and nonexistence of radial solutions to problem (1.1_m).

THEOREM 2.9. *Suppose that* (F2′) *and* (F3) *hold. Then we have*

(i) *If there exists some* $0<\eta_H\leqslant\infty$ *such that* $\bar{H}(u)\equiv 0$ *in* $[0,\eta_H)$, *then every solution* $u(t,\alpha)$, $0<\alpha\leqslant\eta_H$ *is a ground state solution.*

(ii) *If there exists an* $\varepsilon_H>0$ *such that* $\bar{H}(u)$ *is not identically zero in any subinterval of* $[0,\varepsilon_H)$, *and*

$$\bar{H}(u)\geqslant 0 \qquad \text{in } [0,\infty), \tag{2.24}$$

then every solution $u(t, \alpha)$, $\alpha > 0$ is a slowly decaying solution. Consequently, Problem (1.1_m) *admits no radial solutions in any finite ball.*

(iii) *If there exists an $\varepsilon_H > 0$ such that $\bar{H}(u)$ is not identically zero in any subinterval of $[0, \varepsilon_H)$, and*

$$\bar{H}(u) \leqslant 0 \qquad \text{in } [0, \infty), \tag{2.25}$$

then every solution $u(t, \alpha)$, $\alpha > 0$ is a crossing solution.

Proof. (i) Let $u(t, \alpha)$ be a solution with $0 < \alpha < \eta_H$. Then $u(t, \alpha) < \eta_H$ when $0 < t < b(\alpha)$. Now by the assumption of (i) and identity (2.6) we obtain

$$\bar{P}(t, \alpha) \equiv 0, \qquad 0 \leqslant t < b(\alpha).$$

Thus $b(\alpha) = \infty$ and $u(t, \alpha)$ is not a crossing solution. By (2.17) one sees that $u(t, \alpha)$ is not a slowly decaying solution. Therefore $u(t, \alpha)$ must be a ground state solution.

(ii) By (2.6) and (2.24) we have

$$\bar{P}(t, \alpha) \geqslant 0, \qquad 0 \leqslant t < b(\alpha).$$

Hence $b(\alpha) = \infty$ and $u(t, \alpha)$ is a decaying solution. It is easily seen that there is some $t = T_\varepsilon \geqslant 0$ such that

$$0 < u(t, \alpha) < \varepsilon_H, \qquad \text{for} \quad t > T_\varepsilon,$$

which implies $\bar{P}_\alpha > 0$. Hence $u(t, \alpha)$ is a slowly decaying solution.

(iii) The proof is similar to that of (ii). We omit it. ∎

3. PROOF OF THE MAIN THEOREM

In this section, we prove the main result of this paper. We assume (F1) holds, $n \geqslant 3$ and $1 < m < n$, throughout this section. The main theorem is an immediate consequence of the following two technical lemmas.

LEMMA 3.1. *Suppose that*

$$H'(u) \leqslant 0 \qquad \text{in } (0, \infty). \tag{3.1}$$

If $0 < \alpha_1 < \alpha_2$, and $u_1 = u(t, \alpha_1)$, $u_2 = u(t, \alpha_2)$ are two crossing solutions of Problem (2.1_m) with $b(\alpha_1) = b(\alpha_2)$, then $u_1 \leqslant u_2$ in $[0, b(\alpha_1)]$.

LEMMA 3.2. *Assume that $f(u)$ satisfies* (F2). *If $u_1 = u(t, \alpha_1)$ and $u_2 = u(t, \alpha_2)$ are two crossing solutions of Problem* (2.1_m) *with $b(\alpha_1) = b(\alpha_2)$ and $u_1 \leqslant u_2$ in $[0, b(\alpha_1)]$, then $u_1 \equiv u_2$.*

It remains to prove Lemmas 3.1–3.2. We provide a lemma before proving Lemma 3.1.

LEMMA 3.3. *Let $t_1 = t(u, \alpha_1)$, $t_2 = t(u, \alpha_2)$ be the inverses of $u_1(t, \alpha)$ and $u_2(t, \alpha)$, respectively. Define*

$$S_{12}(u) = \frac{t_1^{n-1}}{t_1' \, |t_1'|^{m-2}} \bigg/ \frac{t_2^{n-1}}{t_2' \, |t_2'|^{m-2}}. \tag{3.2}$$

Then

$$S_{12}'(u) > 0 \qquad \textit{if and only if} \quad t_1'(u) > t_2'(u), \quad u \in (0, \alpha_1). \tag{3.3}$$

Proof. We have

$$\begin{aligned}
(t_1^{1-n} t_1' \, |t_1'|^{m-2})' &= (1-n) \, t_1^{-n} t_1'^2 \, |t_1'|^{m-2} + t_1^{1-n} (m-1) \, |t_1'|^{m-2} \, t_1'' \\
&= t_1^{1-n} \, |t_1'|^{m-2} \left(\frac{1-n}{t_1} \, t_1'^2 + (m-1) \, t_1'' \right) \\
&= t_1^{1-n} t_1' \, |t_1'|^{2m-2} \, f(u),
\end{aligned}$$

and a similar identity holds for t_2. Hence

$$\begin{aligned}
\frac{dS_{12}(u)}{du} &= (t_1^{1-n} t_1' \, |t_1'|^{m-2})^{-2} \, [t_2^{1-n} t_2' \, |t_2'|^{2m-2} \, f(u) \, t_1^{1-n} t_1' \, |t_1'|^{m-2} \\
&\quad - t_1^{1-n} t_1' \, |t_1'|^{2m-2} \, f(u) \, t_2^{1-n} t_2' \, |t_2'|^{m-2}] \\
&= (t_1^{1-n} t_1' \, |t_1'|^{m-2})^{-2} \, t_1^{1-n} t_2^{1-n} t_1' \, |t_1'|^{m-2} \\
&\quad \times t_2' \, |t_2'|^{m-2} \, f(u) (|t_2'|^m - t_1'|^m).
\end{aligned}$$

Now (3.3) readily follows. ∎

Proof of Lemma 3.1. Let $t_1 = t(u, \alpha_1)$, $t_2 = t(u, \alpha_2)$ be the inverses of $u_1(t, \alpha)$ and $u_2(t, \alpha)$, respectively. t_i is defined in $[0, \alpha_i]$, $i = 1, 2$. If the assertion of this lemma is not valid, then the graphs of t_1 and t_2 intersect in $(0, \alpha_1)$, and there is some $u_I \in (0, \alpha_1)$ such that

$$t_1(u_I) = t_2(u_I), \quad t_1'(u_I) < t_2'(u_I), \qquad \text{and} \qquad t_1(u) < t_2(u) \quad \text{in } (u_I, \alpha_1). \tag{3.4}$$

But $t_1(0)=t_2(0)=b_1(\alpha)$. It follows that there is a point $u_{II}\in[0,u_I)$ such that

$$t_1(u_{II})=t_2(u_{II}), \qquad \text{and} \qquad t_1(u)>t_2(u) \quad \text{in } (u_{II},u_I).$$

In particular, one has

$$t_1'(u_{II})>t_2'(u_{II}).$$

Thus one can find $u_c\in(u_{II},u_I)$ such that

$$t_1(u_c)>t_2(u_c), \qquad t_1'(u_c)=t_2'(u_c),$$

and

$$t_1'(u)<t_2'(u) \qquad \text{for} \quad u\in(u_c,u_I). \tag{3.5}$$

Moreover, as shown in Lemma 3.3.1 of [9], we have

$$t_1'(u)<t_2'(u) \qquad \text{for} \quad u\in(u_I,\alpha_1). \tag{3.6}$$

Let $S_c=S_{12}(u_c)$, where $S_{12}(u)$ is defined in (3.2), then

$$1<S_c=\frac{t_1^{n-1}(u_c)}{t_2^{n-1}(u_c)}<\frac{t_1^n(u_c)}{t_2^n(u_c)}. \tag{3.7}$$

Let $P_i(u)$, $i=1,2$, denote the corresponding functions of (1.12) when t and t' in (1.12) are replaced with t_i, t_i'. Then

$$\begin{aligned} P_1(u_c)-S_cP_2(u_c) &= [H(u_c)-(n-m)u_c] \\ &\quad \times\left(\frac{t_1^{n-1}(u_c)}{t_1'(u_c)\,|t_1'(u_c)|^{m-2}}-S_c\frac{t_2^{n-1}(u_c)}{t_2'(u_c)\,|t_2'(u_c)|^{m-2}}\right) \\ &\quad -(m-1)\left[\frac{t_1^n(u_c)}{|t_1'(u_c)|^m}-S_c\frac{t_2^n(u_c)}{|t_2'(u_c)|^m}\right] \\ &\quad -mF(u_c)(t_1^n(u_c)-S_ct_2^n(u_c)) \\ &= \left[-\frac{m-1}{|t_1'(u_c)|^m}-mF(u_c)\right][t_1^n(u_c)-S_ct_2^n(u_c)] \\ &< 0. \end{aligned} \tag{3.8}$$

Since $m>1$, $F(u_c)>0$, and (3.7) is satisfied. By using identity (1.11), we have

$$\begin{aligned}
P_1(u_c) - S_c P_2(u_c) &= \int_{\alpha_1}^{u_c} H'(\tau)\, t_1^{n-1}(\tau)/(t_1'(\tau)\, |t_1'(\tau)|^{m-2})\, d\tau \\
&\quad - S_c \int_{\alpha_2}^{u_c} H'(\tau)\, t_2^{n-1}(\tau)/(t_2'(\tau)\, |t_2'(\tau)|^{m-2})\, d\tau \\
&= \int_{\alpha_1}^{u_c} H'(\tau) [\, t_1^{n-1}(\tau)/(t_1'(\tau)\, |t_1'(\tau)|^{m-2}) \\
&\quad - S_c t_2^{n-1}(\tau)/(t_2'(\tau)\, |t_2'(\tau)|^{m-2})]\, d\tau \\
&\quad - S_c \int_{\alpha_2}^{\alpha_1} H'(\tau)\, t_2^{n-1}(\tau)/(t_2'(\tau)\, |t_2'(\tau)|^{m-2})\, d\tau \\
&= I_1 - I_2. \qquad (3.9)
\end{aligned}$$

Since $\alpha_1 < \alpha_2$, $H'(\tau) \leqslant 0$ in (α_1, α_2), and $t_2'(u) < 0$ in (α_1, α_2), we have

$$I_2 \leqslant 0. \qquad (3.10)$$

By (3.5), (3.6) and Lemma 3.3, we know that $S_{12}(u)$ is strictly decreasing in (u_c, α_1). Hence

$$S_{12}(u) < S_c, \qquad \text{for} \quad u \in (u_c, \alpha_1).$$

That is,

$$\frac{t_1^{n-1}(u)}{t_1'(u)\, |t_1'(u)|^{m-2}} - S_c \frac{t_2^{n-1}(u)}{t_2'(u)\, |t_2'(u)|^{m-2}} > 0, \qquad u \in (u_c, \alpha_1). \qquad (3.11)$$

Combining (3.11), and the fact that $u_c < \alpha_1$ and $H'(\tau) \leqslant 0$, we get

$$I_1 \geqslant 0. \qquad (3.12)$$

Therefore,

$$P_1(u_c) - S_c P_2(u_c) \geqslant 0,$$

which contradicts (3.2). The proof is completed. ∎

As an immediate consequence of the proof of Lemma 3.1, we have

Corollary 3.5. *Suppose that* (3.1) *holds. Let* $u_1 = u(t, \alpha_1)$ *and* $u_2 = u(t, \alpha_2)$ *with* $0 < \alpha_1 < \alpha_2$. *Then the graphs of* u_1 *and* u_2 *intersect at most once in* $[0, \min\{b(\alpha_1), b(\alpha_2)\})$.

The proof of Lemma 3.2 is standard. It makes use of the variational property of the first eigenvalue of quasilinear elliptic equations. The proof we give below follows essentially from that of Adimurthi and Yadava (see Lemma 4.1 of [1]).

Proof of Lemma 3.2. It follows from (F2) that the function $\bar{f}(u) = f(u)/u^{m-1}$ is strictly increasing for $u > 0$, and $\bar{f}_0 = \lim_{u \to 0^+} \bar{f}(u)$ exists and is nonnegative. Let

$$\rho_i(t) = f(u_i(t))/u_i^{m-1}(t), \qquad 0 \leqslant t < \bar{b}, \quad \rho_i(\bar{b}) = \bar{f}_0, \quad i = 1, 2,$$

where $\bar{b} = b(\alpha_1) = b(\alpha_2)$. Then $\rho_i(t)$, $i = 1, 2$ are continuous and nonnegative in $[0, \bar{b}]$ and

$$\rho_1(t) = \rho_2(t) \Leftrightarrow u_1(t) = u_2(t), \qquad t \in [0, \bar{b}]. \tag{3.13}$$

Since $u_1 \leqslant u_2$ in $[0, \bar{b}]$, we have

$$\rho_1(t) \leqslant \rho_2(t), \qquad t \in [0, \bar{b}]. \tag{3.14}$$

Note that as u_i is a solution of (2.1_m), $(1, u_i)$ is an eigenpair of the eigenvalue problem

$$\begin{gathered} -(t^{n-1} |\phi'|^{m-2} \phi')' = \lambda \rho_i(t) |\phi|^{m-2} \phi t^{n-1}, \qquad t \in [0, \bar{b}], \\ \phi'(0) = \phi(\bar{b}) = 0. \end{gathered} \tag{3.15}$$

It was proved by Otani and Teshima (see Theorem 1 of [20]) that *the eigenvalue problem* (3.15) *has a nontrivial nonnegative solution* ϕ_i *if and only if* $\lambda = \lambda_{1i}$, *the first eigenvalue, and*

$$\lambda_{1i} = \min_{v \in E_{\bar{b}}} \frac{\int_0^{\bar{b}} |v'|^m t^{n-1} dt}{\int_0^{\bar{b}} \rho_i(t) |v|^m t^{n-1} dt},$$

where $E_{\bar{b}} = \{u \in C^1[0, \bar{b}] \mid u'(0) = u(\bar{b}) = 0\}$. Thus,

$$\begin{aligned} 1 &= \frac{\int_0^{\bar{b}} |u_1'|^m t^{n-1} dt}{\int_0^{\bar{b}} \rho_1(t) |u_1|^m t^{n-1} dt} = \frac{\int_0^{\bar{b}} |u_2'|^m t^{n-1} dt}{\int_0^{\bar{b}} \rho_2(t) |u_2|^m t^{n-1} dt} \\ &\leqslant \frac{\int_0^{\bar{b}} |u_1'|^m t^{n-1} dt}{\int_0^{\bar{b}} \rho_2(t) |u_1|^m t^{n-1} dt}, \end{aligned}$$

which implies

$$\int_0^{\bar{b}} \rho_1(t) |u_1|^m t^{n-1} dt \geqslant \int_0^{\bar{b}} \rho_2(t) |u_1|^m t^{n-1} dt. \tag{3.16}$$

Since $u_1 > 0$ in $[0, \bar{b})$, (3.14) and (3.16) imply $\rho_1(t) \equiv \rho_2(t)$, $t \in [0, \bar{b})$. Therefore, $u_1 \equiv u_2$. The proof is completed. ∎

4. APPLICATIONS OF THE MAIN THEOREM

In this section, we apply the main theorem to the uniqueness problem of (1.1) and (1.1_m) when f is of canonical forms $f(u)=u^p$, $p>0$ and $f(u)=\lambda u^p+u^q$, $\lambda>0$, $p<q$. In particular, Theorem 1 can be proved.

First we consider the simpler case $f(u)=u^p$. (F2) is fulfilled when $p>m-1$, while $H(u)=(n-m-nm/(p+1))u$ and $H'(u)=n-m-nm/(p+1)$. Therefore,

$$H'(u)<0 \qquad \text{if and only if} \quad p<\frac{nm-n+m}{n-m}.$$

The following result follows immediately from the main theorem and Theorem 2.9.

THEOREM 4.1. *Suppose that $f(u)=u^p$, $p>m-1$. Then we have*

(i) *if p is "supercritical", i.e.,*

$$p>\frac{nm-n+m}{n-m},$$

then problem (1.1_m) has no radial solution. Moreover, every solution $u(t,\alpha)$, $\alpha>0$ is a slowly decaying solution.

(ii) *If p is "critical", i.e.,*

$$p=\frac{nm-n+m}{n-m},$$

then Problem (1.1_m) has no radial solution. Moreover, every solution $u(t,\alpha)$. $\alpha>0$ is a ground state solution.

(iii) *If p is "subcritical", i.e.,*

$$p<\frac{nm-n+m}{n-m},$$

then every solution $u(t,\alpha)$, $\alpha>0$ is a crossing solution, and Problem (1.1_m) admits at most one radial solution.

Next, we consider the problem of uniqueness of radial solutions u satisfying

$$\begin{gathered}\operatorname{div}(|\nabla u|^{m-2}\nabla u)+\lambda u^p+u^q=0 \quad \text{in } B,\\ u>0 \quad \text{in } B, \qquad u=0 \quad \text{on } \partial B,\end{gathered} \tag{4.1}$$

where $\lambda > 0$ is a constant, and $1 < m < n$. When $m = 2$, (4.1) reduces to the Dirichlet problem for the semilinear elliptic equation,

$$\begin{gathered} \Delta u + \lambda u^p + u^q = 0 \quad \text{in } B, \\ u > 0 \quad \text{in } B, \qquad u = 0 \quad \text{on } \partial B. \end{gathered} \tag{4.2}$$

We are only concerned here with the case when the nonlinearity is superlinear and subcritical (or critical). We may assume, corresponding to (4.2), that

$$1 \leqslant p < q \leqslant \frac{n+2}{n-2}. \tag{4.3}$$

While for the m-Laplacian equation (4.1), we assume that

$$m - 1 \leqslant p < q \leqslant \frac{nm - n + m}{n - m}. \tag{4.4}$$

Note that if $1 < m < n$, then we always have

$$m - 1 < \frac{nm - n + m}{n - m},$$

which shows that (4.4) is feasible.

Let $f(u) = \lambda u^p + u^q$, then

$$F(u) = \frac{\lambda}{p+1} u^{p+1} + \frac{1}{q+1} u^{q+1},$$

and

$$\begin{aligned} H(u) &= [(n-m)\, u f(u) - nmF(u)]/f(u) \\ &= \bigg[\lambda(n-m)\, u^{p+1} + (n-m)\, u^{q+1} \\ &\qquad - \frac{\lambda nm}{p+1} u^{p+1} - \frac{nm}{q+1} u^{q+1} \bigg] \bigg/ (\lambda u^p + u^q) \\ &= \left[\lambda \left(n - m - \frac{nm}{p+1} \right) u^{p+1} + \left(n - m - \frac{nm}{q+1} \right) u^{q+1} \right] \bigg/ (\lambda u^p + u^q). \\ &= \left[\lambda \left(n - m - \frac{nm}{p+1} \right) u + \left(n - m - \frac{nm}{q+1} \right) u^{q-p+1} \right] \bigg/ (\lambda + u^{q-p}). \end{aligned}$$

For simplicity of notations, we let

$$\nu = -\left(n - m - \frac{nm}{p+1}\right), \qquad \xi = -\left(n - m - \frac{nm}{q+1}\right). \tag{4.5}$$

Then

$$H(u) = -\frac{\lambda \nu u + \xi u^{q-p+1}}{\lambda + u^{q-p}}. \tag{4.6}$$

It can be easily verified that when (4.4) is satisfied, $\nu > 0$, $\xi \geqslant 0$ and $H(u) < 0$, for $u > 0$. Thus every solution $u(t, \alpha)$, $\alpha > 0$, has a finite zero. Differentiating (4.6) with respect to u yields

$$H'(u) = -\frac{1}{(\lambda + u^{q-p})^2} [\lambda^2 \nu + \lambda[\nu(p - q + 1) + \xi(q - p + 1)] v + \xi v^2],$$

where $v = u^{q-p}$. Let

$$\sigma = \nu(p - q + 1) + \xi(q - p + 1), \tag{4.7}$$

and

$$D(x) = D(x;\, p,\, q,\, \lambda) = \xi x^2 + \lambda \sigma x + \lambda^2 \nu. \tag{4.8}$$

Then

$$H'(u) = -\frac{1}{(\lambda + u^{q-p})^2} \cdot D(v). \tag{4.9}$$

Applying the main theorem, we have

THEOREM 4.2. *Suppose that* $\lambda > 0$, *p and q satisfy* (4.4). *Let* $D(x)$ *be the quadratic form of* (4.8). *Then problem* (4.1) *has at most one radial solution if*

$$D(x) > 0, \qquad \text{for} \quad x > 0, \tag{4.10}$$

or equivalently, if

$$\sigma \geqslant 0 \qquad \text{or} \quad \sigma^2 \leqslant 4\nu\xi, \tag{4.11}$$

where ν, ξ *and* σ *are defined in* (4.5) *and* (4.7).

Since $v > 0$, $\xi \geqslant 0$ and $p < q$, we see that $p - q + 1 \geqslant 0$, i.e.,

$$q - p \leqslant 1, \tag{4.12}$$

which implies $\sigma \geqslant 0$.

COROLLARY 4.3. *Suppose that* $\lambda > 0$, *and* p *and* q *satisfy* (4.4). *Then problem* (4.1) *possesses at most one radial solution provided that* $q - p \leqslant 1$.

COROLLARY 4.4. *Suppose that* $\lambda > 0$, *and* p *and* q *satisfy* (4.4). *Then Problem* (4.1) *possesses at most one radial solution when* $n \geqslant m + m^2$.

Proof. It suffices to show that $n \geqslant m + m^2$ implies $q - p \leqslant 1$. In fact, by (4.4) we obtain

$$q - p \leqslant \frac{nm - n + m}{n - m} - (m - 1) = \frac{m^2}{n - m} \leqslant \frac{m^2}{m + m^2 - m} = 1. \quad \blacksquare$$

Proof of Theorem 1. The existence of a solution u to Problem (1.5) was proved in [3]. It follows from [8] that u is radial. The uniqueness of u follows from Corollary 4.4. ∎

Some other uniqueness results to Problems (4.2)–(4.3) can be derived for the remaining interesting cases $n = 3$, 4, 5, by using Corollary 4.3.

Remark 4.5. When $n = 3$, we have

$$v = \frac{5 - p}{p + 1}, \qquad \xi = \frac{5 - q}{q + 1},$$

and

$$\sigma = \frac{2}{(p + 1)(q + 1)} (5pq + 2p + 2q - 3p^2 - 3q^2 + 5).$$

Condition (4.11) becomes

$$5pq + 2p + 2q - 3p^2 - 3q^2 + 5 \geqslant 0, \tag{4.13}$$

or

$$(5pq + 2p + 2q - 3p^2 - 3q^2 + 5)^2 \leqslant (5 - p)(5 - q)(p + 1)(q + 1). \tag{4.14}$$

It follows from Theorem 4.2 that *if either* (4.13) *or* (4.14) *holds, then nonlinear Problem* (4.2)–(4.3) *with* $n = 3$ *admits at most one solution.*

Some similar inequalities to (4.13) and (4.14) can be established when $n = 4$ or $n = 5$. We omit the details.

EXAMPLE 4.6. *The case* $n = 3$ *and* $p = 3$: Inequalities (4.13) and (4.14) can be simplified to $3q^2 - 17q + 16 \leqslant 0$ and

$$9q^4 - 102q^3 + 292q^2 - 576q + 216 \leqslant 0. \tag{4.15}$$

It can be verified that (4.15) is satisfied when $3 < p < 4.7748332$. Therefore, *the semilinear Dirichlet problem,*

$$\Delta u + \lambda u^3 + u^q = 0 \quad \text{on } B_1,$$
$$u > 0 \quad \text{on } B_1, \qquad u = 0 \quad \text{on } \partial B_1,$$

has a unique solution when $3 < q < 4.7748332$.

Remark 4.7. Let B_1 be the unit ball in $\mathbb{R}^3$. Consider the problem

$$\begin{gathered} \Delta u + \lambda u^p + u^5 = 0 \quad \text{on } B_1, \\ u > 0 \quad \text{on } B_1, \qquad u = 0 \quad \text{on } \partial B_1. \end{gathered} \tag{4.16}$$

Recall from [2] and [3] that if $1 < p \leqslant 3$ and $\lambda > \lambda_0$ for some positive constant λ_0, then (4.16) has at least two solutions. On the other hand, it follows from Corollary 4.3 that if $4 \leqslant p < 5$, then (4.16) has a unique solution for all $\lambda > 0$. We conjecture that *there is a number* $p = p^*$ *with* $3 \leqslant p^* \leqslant 4$ *such that the uniqueness of solutions of* (4.16) *holds for* $p^* < p < 5$, *and the uniqueness is lost for* $1 < p < p^*$.

Remark 4.8. In the situation that the supercritical growth is involved in (4.2), one may not expect to have the uniqueness. Budd and Norbury [4] proved that, when $n = 3$, $p = 1$ and $q > 5$, there is a critical value $\lambda = \lambda_c(q)$ at which problem (4.2) has infinitely many positive C^2 solutions.

5. NONUNIQUENESS-MULTIPLE ORDERED SOLUTIONS

Throughout this section we restrict ourselves to the semilinear problem (1.1) and study the multiplicity of positive solutions. When $m = 2$, (F2) is reduced to the requirement that $f(u)$ is superlinear, i.e., $f(u)/u$ is an increasing function of u in $u > 0$. In this case, the main theorem can be stated as: *if f is superlinear and $H'(u) < 0$ in $u > 0$, then the semilinear elliptic Problem* (1.1) *has at most one solution*, while if f is sublinear, i.e.,

$$uf'(u) < f(u) \qquad \text{for} \quad u > 0, \tag{5.1}$$

then we can easily demonstrate the uniqueness of positive solutions to (1.1) even without assuming $H'(u) < 0$ in $u > 0$.

In what follows we shall consider the existence of multiple solutions of Problem (1.1) with

$$f(u) = u^p - u^q, \qquad 1 < p < \frac{n+2}{n-2} \leqslant q. \tag{5.2}$$

This function has both superlinear and sublinear growth in $(0, \infty)$.

PROPOSITION 5.1. *Let f be defined in* (5.2). *Let $u(t, \alpha)$ be a solution of* (2.1). *We have*

(i) *for $\alpha > 1$, $u(t, \alpha)$ is positive and strictly increasing in $(0, \zeta(\alpha))$, where $0 < \zeta(\alpha) \leqslant \infty$, and $\lim_{t \to \zeta(\alpha)} u(t, \alpha) = \infty$ when $\zeta(\alpha) < \infty$.*

(ii) $u(t, 1) \equiv 1$.

(iii) *for $0 < \alpha < 1$, $u(t, \alpha)$ is a crossing solution and is strictly decreasing before it vanishes.*

Proof. (i) If $\alpha > 1$, then $u(t, \alpha) > 1$ in a right neighborhood of $t = 0$. Let $t = \bar{t} > 0$ be a point such that $u(t, \alpha) > 1$ in $[0, \bar{t}]$. It suffices to prove that $u'(\bar{t}, \alpha) > 0$. In fact, from the equation of (2.1) one can derive

$$\bar{t}^{n-1} u'(\bar{t}, \alpha) = -\int_0^{\bar{t}} t^{n-1} f(u)\, dt > 0,$$

and therefore $u'(\bar{t}, \alpha) > 0$.

(ii) It follows from the fact that $f(1) = 0$.

(iii) Observe that $f(u) > 0$ when $0 < u < 1$. Thus if $0 < \alpha < 1$, then $u(t, \alpha)$ is strictly decreasing whenever it is positive. We only need to prove that $u(t, \alpha)$ has a finite zero. Since for function $f(u)$ in (5.2),

$$H(u) = \frac{1}{f(u)} \left[\left(n - 2 - \frac{2n}{p+1} \right) u^{p+1} - \left(n - 2 - \frac{2n}{q+1} \right) u u^{q+1} \right], \tag{5.3}$$

we see that $H(u) < 0$ when $0 < u < 1$. It follows from Theorem 2.9 that $u(t, \alpha)$ is a crossing solution when $0 < \alpha < 1$. The proof is completed. ∎

When $\alpha \geqslant 1$, this proposition shows that $u(t, \alpha)$ is neither a crossing solution nor a decaying solution. Thus our consideration below can be restricted in $0 < u < 1$. We say that *two positive solutions $u_1(x)$ and $u_2(x)$ of* (1.1) *are strictly ordered if*

$$u_1(x) < u_2(x), \qquad x \in B. \tag{5.4}$$

The main result of this section is

THEOREM 5.2. *Let $f(u)$ be as in* (5.2). *Let $R>0$ be the radius of B. Then there exists $\bar{R}>0$ such that*

(i) *if $R<\bar{R}$, then Problem* (1.1) *has no solutions*;

(ii) *if $R>\bar{R}$, then Problem* (1.1) *admits at least two solutions, and any two distinct solutions are strictly ordered.*

Proof. First we prove the second part of (ii). In view of Corollary 3.4 and Proposition 5.1, we only need to show that $H'(u)<0$ in $0<u<1$. Differentiating (5.3) we obtain

$$H'(u)=\frac{u^{2p}}{f^2(u)}\cdot T(v), \tag{5.5}$$

where $v=u^{q-p}$, and

$$T(v)=\left(n-2-\frac{2n}{q+1}\right)v^2 + \left[2n+4-2n\left(\frac{p}{q+1}+\frac{q}{p+1}\right)\right]v+\left(n-2-\frac{2n}{p+1}\right).$$

Since $T(0)=n-2-2n/(p+1)<0$, $T''(v)\equiv 2(n-2-2n/(q+1))\geqslant 0$, and

$$T(1)=-\frac{2n}{(p+1)(q+1)}(p-q)^2<0.$$

We conclude that $T(v)<0$ when $0\leqslant v\leqslant 1$. Hence

$$H'(u)<0 \qquad \text{in} \quad 0<u<1, \tag{5.6}$$

as desired.

It follows from Proposition 5.1 that when $0<\alpha<1$, $b(\alpha)$ is defined and $b(\alpha)<\infty$. The existence and uniqueness theorem for initial value problems for ordinary differential equations implies that $u'(b(\alpha),\alpha)<0$. Thus, combining the fact that $u(t,\alpha)$ is C^2 in t and C^1 in α, we see that $b(\alpha)$ is continuous. Since $u(t,1)\equiv 1$. The continuity of $b(\alpha)$ implies that $b(\alpha)\to +\infty$ as $\alpha\to 1^-$.

We claim that if $\alpha\to 0^+$, then it also holds that $b(\alpha)\to +\infty$. Assuming the claim for the moment, we can readily complete the proof of this theorem. In fact, if $b(\alpha)\to +\infty$ as $\alpha\to 0^+$ or $\alpha\to 1^-$, then ($b(\alpha)$ attains its absolute minimum at some points, say, at $\alpha=\bar{\alpha}$, with $0<\bar{\alpha}<1$, and

$$b(\bar{\alpha})=\inf_{0<\alpha<1}\{b(\alpha)\}, \qquad 0<\bar{\alpha}<1.$$

Obviously, $b(\bar{\alpha}) > 0$. Let $\bar{R} = b(\bar{\alpha})$. Then problem (1.1) has no solution if $R < \bar{R}$ and (i) is proved. The first part of (ii) follows from the continuity of $b(\alpha)$ and the mean value theorem.

Now we turn to the proof of the claim. In order to investigate the behavior of solutions with sufficiently small initial data, we use a standard scaling argument. The proof we present next is essentially due to Ni and Yotsutani [19].

For a given solution $u = u(t, \alpha)$, let $w = w(t, \alpha) = (1/\alpha)\, u(t/\beta, \alpha)$. Then

$$w(0) = 1, \qquad w'(0) = 0, \tag{5.7}$$

and

$$w'' + \frac{n-1}{t} w' + \frac{\alpha^{p-1}}{\beta^2} w^p - \frac{\alpha^{q-1}}{\beta^2} w^q = 0.$$

If $\beta = \alpha^{(p-1)/2}$, then

$$w'' + \frac{n-1}{t} w' + w^p - \alpha^{q-p} w^q = 0. \tag{5.8}$$

Let $v(t)$ be the unique solution of problem

$$\begin{aligned} & v'' + \frac{n-1}{t} v' + v^p = 0, \\ & v(0) = 1, \qquad v'(0) = 0. \end{aligned} \tag{5.9}$$

Then $v'(t) < 0$ as long as $v(t) > 0$. Let t_v be the unique point at which $v(t_v) = 1/2$. Then

$$v(t) \geqslant \tfrac{1}{2} \qquad \text{in } [0, t_v].$$

Note that t_v does exist in view of (v) of Proposition 2.1. We claim that

$$\lim_{\alpha \to 0^+} \Big[\sup_{0 \leqslant t \leqslant t_v} |w - v| \Big] = 0 \tag{5.10}$$

If this claim is proved, then for α sufficiently close to 0, one has

$$w(t, \alpha) = \frac{1}{2} u\left(\frac{t}{\beta}, \alpha\right) > 0, \qquad t \in [0, t_v].$$

Therefore, $b(\alpha) > t_v/\beta$, which implies that $b(\alpha) \to +\infty$ as $\alpha \to 0^+$ as desired.

It remains to prove (5.10). It can be easily verified that (see [19])

$$v(t)=1-\frac{1}{n-2}\int_0^t\left[1-\left(\frac{s}{t}\right)^{n-2}\right]sv^p(s)\,ds,$$

$$w(t)=1-\frac{1}{n-2}\int_0^t\left[1-\left(\frac{s}{t}\right)^{n-2}\right][w^p(s)-\alpha^{q-p}w^q(s)]\,ds.$$

Taking the difference of $v(t)$ and $w(t)$, we obtain

$$\begin{aligned}(n-2)\,|w(t)-v(t)|&=\left|\int_0^t\left[1-\left(\frac{s}{t}\right)^{n-2}\right]s[v^p-w^p+\alpha^{q-p}w^q]\,ds\right|\\&\leqslant\int_0^t s\,|v^p-w^p|\,dt+\alpha^{q-p}\int_0^t sw^q\,ds.\end{aligned}$$

Thus, for $0\leqslant t\leqslant t_v$, one has $|v(t)|\leqslant 1$, $|w(t)|\leqslant 1$, and

$$|w(t)-v(t)|\leqslant\frac{1}{n-2}\cdot\alpha^{q-p}\cdot\frac{t_v^2}{2}+\frac{p}{n-2}\int_0^t s\,|v-w|\,dt.$$

By the well-known Gronwall's inequality, we obtain

$$\sup_{0\leqslant t\leqslant t_v}|w(t)-v(t)|\leqslant\frac{1}{n-2}\cdot\alpha^{q-p}\cdot\frac{t_v^2}{2}\cdot\exp\left(\frac{p}{n-2}\cdot\frac{t_v^2}{2}\right),$$

which implies $\lim_{\alpha\to 0^+}\sup_{0\leqslant t\leqslant t_v}|w(t)-v(t)|=0$. The proof is completed. ∎

Theorem 5.2 can be easily extended to some more general nonlinearities $f(u)$. But a more interesting problem is to obtain the exact multiplicity of solutions whose existence is ensured by this theorem. We conjecture that the number of critical points of $b(\alpha)$ in $(0, 1)$ is one, and there exist exactly two solutions when $R>\bar{R}$. For the study of exact multiplicity problem, there are few results in the literature. Ouyang [21] considered the structure of positive solutions of semilinear equations $\Delta u+\lambda u+hu^p=0$ on compact manifolds, where $\lambda>0$, $p>1$ are real numbers, and $h=h(x)$ is a real function. Under some natural conditions, he proved that the Dirichlet problem has exactly two positive solutions. But his idea seems not applicable to Problem (1.1) when f is defined in (5.2), since this nonlinearity contains both supercritical (or critical) and subcritical terms. It also has a superlinear growth for $0<u<((p-1)/(q-1))^{1/(q-p)}$, and a sublinear growth for $((p-1)/(q-1))^{1/(q-p)}<u<1$.

ACKNOWLEDGMENT

We are very grateful to Professor James Serrin for his useful advice. In particular, his valuable comments led to a simplified proof of the main theorem.

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